Problem 8.5.3

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02407 Stochastic Processes

In this problem, we need to verify the option pricing formulation in eq. (8.66). The underlying assumption is that the stock price process $\{S_t\}_{t\geq 0}$ is driven by an Ornstein-Uhlenbeck process $\{V_t\}_{t\geq 0}$, i.e.

$$S_t = S_0 + \int_0^t V_u du. \tag{1}$$

We then consider a call option on the stock with strike K and maturity T. The risk-neutral valuation (the fair price) of this option is then given as

$$F(S_0, T) = \mathbb{E}[\max(S_T - K, 0)].$$

When S_t is described as in eq. (1), it follows from p. 443-444 that $S_t \sim \mathcal{N}(S_0, \sigma_t^2)$, where

$$\sigma_t^2 = \frac{\sigma^2}{\beta^2}\left[t - \frac{2}{\beta}\left(1 - e^{-\beta t}\right) + \frac{1}{2\beta}\left(1 - e^{-2\beta t}\right)\right].$$

In the above expression, the parameters β and σ^2 are the drift coefficient and the diffusion parameter of the process $\{V_t\}$, respectively. As we know from the problem that $S_0 = z$, we see that given this initial condition, $S_T \sim \mathcal{N}(z, \sigma_T^2)$. We proceed by applying the result from Exercise 8.4.6 b) to evaluate the fair price;

$$\mathbb{E}[\max(S_T - K, 0)] = \sigma_T\left(\phi\left(\frac{K-z}{\sigma_T}\right) - \left(\frac{K-z}{\sigma_T}\right)\left[1 - \Phi\left(\frac{K-z}{\sigma_T}\right)\right]\right),$$

where ϕ and Φ are the density function and distribution function, respectively, of a random variable having a standard normal distribution. To obtain the expression in eq. (8.66), we note that the strike price of the option is called a, the maturity of the option is called t, while the mean and variance of S_t (the stock price at maturity) are denoted by μ and τ^2. Inserting these new variable names, we get

$$\mathbb{E}[\max(S_t - a, 0)] = \tau\left(\phi\left(\frac{a-\mu}{\tau}\right) - \left(\frac{a-\mu}{\tau}\right)\left[1 - \Phi\left(\frac{a-\mu}{\tau}\right)\right]\right).$$